MCWG Exercise 5.B.3. key by Prof. Kilkenny
PROVE: "A *production set* is *convex* set *iff* the *production function* is *concave.*"

Let $f(\mathbf{z})=q$: $\mathbf{R}^{L-1} \rightarrow \mathbf{R}$ and
$\mathbf{Y} = \{(-\mathbf{z}, q): q \leq f(\mathbf{z})\ \&\ \mathbf{z} \geq 0\}$.

Claim: **Y** is convex iff $f(\mathbf{z})$ is concave.

Let "**Y** is convex" be statement "A" and "$f(\mathbf{z})$ is concave " be statement "B."
We have to show A$\leftrightarrow$B. We will do this in two parts: show A$\Rightarrow$B then show B$\Rightarrow$ A..

<u>First Half of Proof:</u>
By the convexity of **Y**, for all $\mathbf{z}, \mathbf{z}` \subset \mathbf{R}^{L-1}$ such that $(-\mathbf{z}, f(\mathbf{z}))$ & $(-\mathbf{z}`, f(\mathbf{z}`)) \in \mathbf{Y}$, and all $t \in [0,1]$,

$([-(t\mathbf{z}+(1-t)\mathbf{z}`)], tf(\mathbf{z})+(1-t)f(\mathbf{z}`)]) \in \mathbf{Y}$.

By the *transformation function* that defines **Y**, $tf(\mathbf{z})+(1-t)f(\mathbf{z}`) \leq f(t\mathbf{z}+(1-t)\mathbf{z}`)$
which also defines $f(\mathbf{z})$ as concave. In sum, depending on the strictness of the inequalities, if the production set **Y** is (strictly) convex, the production function $f(\mathbf{z})$ is (strictly) concave.

<u>Second half of proof:</u>
By definition of **Y**, $f(\mathbf{z}) \geq q$ & $f(\mathbf{z}`) \geq q`$
The convex combination of the two LHS's is at least as large as the convex combination of the RHS:

$tf(\mathbf{z}) + (1-t) f(\mathbf{z}`) \geq tq + (1-t)q`$.

Let $z'' = t\mathbf{z}+(1-t)\mathbf{z}`$ and $q'' = f(t\mathbf{z}+(1-t)\mathbf{z}`)$ such that $(-\mathbf{z}'', q'') \in \mathbf{Y}$.
The concavity of $f(\mathbf{z}) \Rightarrow f(t\mathbf{z}+(1-t)\mathbf{z}`) \geq tf(\mathbf{z}) + (1-t) f(\mathbf{z}`)$, for all $t \in [0,1]$.

Equivalently, $q'' \geq tq + (1-t)q`$.

Hence $(-(t\mathbf{z}+(1-t)\mathbf{z}`), tf(\mathbf{z})+(1-t)f(\mathbf{z}`)) \in \mathbf{Y}$, which shows that **Y** is convex.
In sum, depending on the strictness of the inequalities, if the production function $f(\mathbf{z})$ is (strictly) concave, the production set **Y** is (strictly) convex.

Taking both halves of the proof together, **Y** is convex iff $f(\mathbf{z})$ is concave.