

MCWG Exercise 5.B.2. key by Prof. Kilkenny

PROVE “A production set satisfies constant returns to scale iff the production function is homogeneous of degree 1.”

“Suppose that $f(\cdot)$ is the production function associated with a single output technology and \mathbf{Y} is the production set of this technology...”

$$f(z) = q: \mathbf{R}^{L-1} \rightarrow \mathbf{R}$$

$$\mathbf{Y} = \{ \mathbf{y} = (-z, q): q \leq f(z) \text{ \& } z > 0 \}$$

Claim: \mathbf{Y} displays CRS iff $f(z)$ is HMG^o1.

Let “ \mathbf{Y} is CRS” be statement “A” and “ $f(z)$ is HMG^o1” be statement “B”
We have to show $A \leftrightarrow B$. We will do this in two parts: show $A \Rightarrow B$ then $B \Rightarrow A$.

First Half: $\mathbf{Y} \text{ CRS} \Rightarrow f(z) \text{ HMG}^{\circ}1$:

$$\mathbf{Y} \text{ CRS} \Rightarrow \forall \mathbf{y} \in \mathbf{Y} \quad \& \alpha \geq 0, \alpha \mathbf{y} \in \mathbf{Y}$$

$$\mathbf{y} = (-z, q) = (-z, f(z)); \quad \alpha \mathbf{y} = (-\alpha z, \alpha q)$$

and by the transformation function that defines \mathbf{Y} , $\alpha q = \alpha f(z) \leq f(\alpha z)$

The production set includes its boundary, so $(-\alpha z, f(\alpha z)) = \mathbf{y}' \in \mathbf{Y}$.

$$\text{By } \mathbf{Y} \text{ CRS}, \forall \mathbf{y}' \in \mathbf{Y} \quad \& \beta \geq 0, \beta \mathbf{y}' \in \mathbf{Y}$$

$$\mathbf{y}' = (-\alpha z, f(\alpha z)); \quad \beta \mathbf{y}' = (-\beta \alpha z, \beta f(\alpha z))$$

and by the transformation function that defines \mathbf{Y} , $\beta f(\alpha z) \leq f(\beta \alpha z)$

$$\text{letting } \beta = 1/\alpha \text{ we have } (1/\alpha) f(\alpha z) \leq f(z)$$

$$\text{multiplying both sides by } \alpha: \quad f(\alpha z) \leq \alpha f(z)$$

Thus we have that $\alpha f(z) \leq f(\alpha z)$

$$\text{and} \quad \alpha f(z) \geq f(\alpha z)$$

which means $\alpha f(z) = f(\alpha z)$

which shows that $f(z)$ is HMG^o1.

In sum, $\mathbf{Y} \text{ CRS} \Rightarrow f(z) \text{ HMG}^{\circ}1$.

Second Half: $f(z) \text{ HMG}^{\circ}1 \Rightarrow \mathbf{Y} \text{ CRS}$

As before, $\mathbf{y} \text{ \& } \mathbf{y}' = (-\alpha z, f(\alpha z)) \in \mathbf{Y}$;

$$f(z) \text{ HMG}^{\circ}1 \Rightarrow \forall \alpha \geq 0, \quad f(\alpha z) = \alpha f(z)$$

and thus $\mathbf{y}' = (-\alpha z, f(\alpha z)) = (-\alpha z, \alpha f(z)) = (-\alpha z, \alpha q) = \alpha \mathbf{y}$, so $\alpha \mathbf{y} \in \mathbf{Y}$
that is, \mathbf{Y} satisfies CRS.

In sum, $f(z) \text{ HMG}^{\circ}1 \Rightarrow \mathbf{Y} \text{ CRS}$.

First + Second Halves together give: $\mathbf{Y} \text{ CRS} \leftrightarrow f(z) \text{ HMG}^{\circ}1$.